10^{th} Lecture

Double Integrals in Polar Coordinates.

Curves and Regions in Polar Coordinates.

There are many important plane curves consisting of the points whose polar coordinates (contrary to Cartesian coordinates) satisfy a simple equation of the form $r = f(\varphi)$.

Example 1. Graph the curve:

1. $r = a(1 - \cos\varphi),\ a > 0$, (cardioid),
2. $r = \sqrt{a\cos 2\varphi},\ a > 0$, (lemniscate),
3. $r = a\varphi,\ a > 0$, (spiral),
4. $r = a|\sin 2\varphi|,\ a > 0$, (four-leaved rose),
5. $r = a|\sin n\varphi|,\ a > 0$, n - natural, ($2n$-leaved rose),
6. $r = a\sin\varphi,\ a > 0$, (a circle).

Domain of integration is often a plane region not bounded by lines, but by curves, like, for instance, arcs of circles. Especially in these cases it is simpler to describe the regions in polar coordinates.

Example 2. Sketch the plane regions, defined in polar coordinates by inequalities:

1. $R = \{[r,\varphi] : 0 \le \varphi \le 2\pi,\ 2 \le r \le 3\}$,
2. $R = \{[r,\varphi] : \pi \le \varphi \le 2\pi,\ 0 \le r \le 1\}$,
3. $R = \{[r,\varphi] : \frac{\pi}{2} \le \varphi \le \pi,\ 0 \le r \le 2\sin\varphi\}$,
4. $R = \{[r,\varphi] : 0 \le \varphi \le \frac{\pi}{2},\ 0 \le r \le 4\cos\varphi\}$,
5. $R = \{[r,\varphi] : \frac{\pi}{2} \le \varphi \le \frac{3\pi}{4},\ 0 \le r \le 4\varphi\}$,
6. $R = \{[r,\varphi] : 0 \le \varphi \le 2\pi,\ 0 \le r \le 1 + \cos\varphi\}$.

Change of Variables in Double Integrals.

The rule for change of variable in the definite integral plays an extremely important role in practical integration. It states: If φ is a function defined on an interval I such that derivative of φ is continuous and different from zero on the interval I and f is a function continuous on $\varphi(I)$, then

$$\int\limits_{\varphi(I)} f(x)dx = \int\limits_{I} f(\varphi(t))\varphi'(t)dt.$$

Change of variables in double integrals is used not only in the case of "too complicated" integrands (functions to be integrated), but also in the case of "too complicated" domains of integration.

The rule for change of variables in the double integrals is essentially more complicated and we shall confine ourselves to the final formula and to the case of the transformation from Cartesian coordinates to polar ones.

If a transformation $\Phi : E_2 \to E_2,\ \Phi[u, v] = [x, y]$ is given by formulas $x = \varphi_1(u, v)$, $y = \varphi_2(u, v)$, where φ_1 and φ_2 are real functions of two variables, then the formula for change of variables in double integrals has the form

$$\iint\limits_{R} f(x, y)dxdy = \iint\limits_{R^*} f(\varphi_1(u, v), \varphi_2(u, v))|D_\Phi(u, v)|dudv,$$

where $R = \Phi(R^*)$ and $D_\Phi(u, v)$ is **Jacobi's functional determinant (Jacobian)**, defined by

$$D_\Phi(u, v) = \begin{vmatrix} \frac{\partial\varphi_1(u,v)}{\partial u}, & \frac{\partial\varphi_1(u,v)}{\partial v} \\ \frac{\partial\varphi_2(u,v)}{\partial u}, & \frac{\partial\varphi_2(u,v)}{\partial v} \end{vmatrix}$$

We shall not present all conditions providing the validity of the formula and only note that the functions $\varphi_1,\ \varphi_2,\ f$ and their partial derivatives are supposed to be continuous.

Let us apply the general formula to the transformation Φ from Cartesian coordinates $[x, y]$ to polar coordinates $[r, \varphi]$, given by formulas

$$x = \varphi_1(r, \varphi) = r\cos\varphi, \qquad y = \varphi_2(r, \varphi) = r\sin\varphi.$$

Then $D_\Phi(r, \varphi) = r$ and therefore

$$\iint\limits_{R} f(x, y)dxdy = \iint\limits_{R^*} f(r\cos\varphi, r\sin\varphi)rd\varphi dr,$$

where $R^* = \Phi^{-1}(R)$ (R and R^* are domains of integration in the planes P_{xy} and $P_{r\varphi}$ with r and φ interpreted as Cartesian coordinates in an auxiliary plane).

Remark 1. If, for example, I is a rectangle in $P_{r\varphi}$ given by

$$I = \{[r, \varphi] : \varphi_1 \le \varphi \le \varphi_2,\ r_1 \le r \le r_2\},$$

for $0 \le \varphi_1 < \varphi_2 \le 2\pi,\ 0 \le r_1 < r_2$, then $\Phi(I)$ is a region (a part of annulus) in P_{xy} bounded by line segments with parametric equations

$$x = r\cos\varphi_1,\ y = r\sin\varphi_1, \quad \text{and} \quad x = r\cos\varphi_2,\ y = r\sin\varphi_2, \qquad r \in \langle r_1, r_2\rangle$$

and by arcs of circles with parametric equations

$$x = r_1 \cos\varphi,\ y = r_1 \sin\varphi, \quad \text{and} \quad x = r_2 \cos\varphi,\ y = r_2 \sin\varphi \qquad \varphi \in \langle \varphi_1, \varphi_2 \rangle.$$

This change of variables may substantially simplify the given integral, for instance, it may lead to constant limits of integration in the transformed integral.

Example 3. By means of the polar transformation compute the integral $\iint\limits_R (x^2+y^2)dxdy$, if $R = \{[x,y] : x^2 + y^2 \le 4\}$.

Example 4. By means of the polar transformation evaluate $\iint\limits_R (x^2 + y^2)dxdy$, if $R = \{[x,y] : 1 \le x^2 + y^2 \le 4,\ y \ge |x|\}$.

Example 5. Compute $\iint\limits_R x dxdy$, if R is a set bounded by the curve $x^2 + y^2 - 2y = 0$.

Applications of Double Integrals in Polar Coordinates.

Remark 2. Although the construction of the region R^* (appearing in the formula for the transformation of double integrals to polar coordinates) in the $P_{r\varphi}$ plane is helpful in understanding the transformation $\Phi : x = r\cos\varphi,\ y = r\sin\varphi$, it is not necessary for determining the limits in the repeated integrals. The limits of integration in polar coordinates may be found by using rectangular and polar coordinates in the same plane and sketch of the original domain R to find out limits for r and φ.

Example 6. By means of polar coordinates find the volume of the solid S, if S is the set

1. bounded by the surfaces $z = 0,\ 2z = x^2 + y^2,\ x^2 + y^2 = 4$,
2. bounded by the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$,
3. cut from the sphere of radius 4 by a cylinder of radius 2 whose axis is a diameter of the sphere,
4. bounded by the cone $z = x^2 + y^2$ and the cylinder $x^2 + y^2 - 2y = 0$,
5. bounded by the cone $z = x^2 + y^2$ and the paraboloid $3z = x^2 + y^2$,
6. bounded by the surfaces $z = x$ and $2z = x^2 + y^2$,
7. bounded by the surfaces $x^2 + y^2 - 2x = 0,\ 4z = x^2 + y^2$ and $z^2 = x^2 + y^2$.

Example 7. By means of polar coordinates find the area of the region R, if R is the region

1. inside the circle $x^2 + y^2 - 8y = 0$ and outside the circle $x^2 + y^2 = 9$,
2. enclosed by the cardioid $r = 1 + \cos\varphi$,
3. inside the circle $x^2 + y^2 - 8y = 0$ and outside the circle $x^2 + y^2 = 9$,
4. bounded by the cone $z = x^2 + y^2$ and the cylinder $x^2 + y^2 - 2y = 0$,
5. bounded by the cone $z = x^2 + y^2$ and the paraboloid $3z = x^2 + y^2$,
6. bounded by the surfaces $z = x$ and $2z = x^2 + y^2$,
7. bounded by the surfaces $x^2 + y^2 - 2x = 0,\ 4z = x^2 + y^2$ and $z^2 = x^2 + y^2$.

Example 8. By means of polar coordinates find the area of the surface S, (the surface area), if S is the portion

1. of the paraboloid $2z = x^2 + y^2$, that is inside the cylinder $x^2 + y^2 = 1$,
2. of the paraboloid $z = 1 - x^2 - y^2$, that is above the plane P_{xy},
3. of the cone $z = \sqrt{x^2 + y^2}$, that lies inside the cylinder $x^2 + y^2 = 2x$,
4. of the surface $z = xy$, that is in the first octant, enclosed by planes $y = 0,\ y = x$, and the cylinder $x^2 + y^2 = 9$.

Example 9 Find the total mass of the semicircular region R bounded by O_x and the curve $y = \sqrt{1 - x^2}$, if the density is $\sigma(x, y) = e^{x^2+y^2}$.

Example 10. A lamina with the density $\sigma(x, y) = xy$ is in the first quadrant and it is bounded by the circle $x^2 + y^2 = 9$ and by the coordinate axes. Find its mass and centre of mass.

Example 11. Find the centroid of the region R, if

1. R is the region, enclosed by the cardioid $r = 1 + \cos\varphi$,
2. R is above O_x and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$,
3. R is enclosed by O_y and the curve $x = \sqrt{9 - y^2}$.