9^{th} Lecture

Geometric and Physical Applications of Double Integrals, The Polar Coordinate System.

Geometric Applications.

Although we do not introduce an exact proof, it is easy to visualize geometrically, that if $f(x, y)$ is a nonnegative (continuous)function defined on a region $R \subset E_2$, then the double integral $\iint\limits_R f(x, y)\ dxdy$ represents the volume of the solid S bounded above by the surface $z = f(x, y)$, below by the plane P_{xy} and laterally by a cylindrical surface generated by vertical lines passing through all points of the boundary of R. Therefore

$$V(S) = \iint\limits_R f(x, y)\ dxdy.$$

Example 1. Find the volume of the tetrahedron bounded by the coordinate planes and by the plane $z = 4 - 4x - 2y$.

Example 2. Find the volume of the solid in the first octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$ and laterally by surfaces $y = x^2$ and $y = x$.

Example 3. Find the volume of the solid enclosed by surface $y^2 = x$ and by planes $z = 0$ and $x + z = 1$.

Example 4. Find the volume of the solid in the first octant bounded

1. by the three coordinate planes and the planes $x + 2y = 4$ and $x + 8y - 4z = 0$,
2. by the surface $z = e^{y-x}$, the plane $x + y = 1$ and the coordinate planes,
3. above by the surface $z = 9 - x^2$, below by $z = 0$ and laterally by the surface $y^2 = 3x$,
4. by the coordinate planes, the plane $x = 3$ and by the surface $z = 4 - y^2$.

In spite of the fact that double integrals arose in the context of calculating volumes, they can also be used to calculate areas. For this purpose we consider the solid consisting of points between the plane $z = 1$ and a region R in the plane P_{xy}.

The volume V of this solid is $(f(x,y)=1): V=\iint\limits_R 1\,dxdy$ and simultaneously this volume equals to the product of the area of the region and the "height" of this solid, $V=A(R)\times 1$, it follows that

$$A(R)=\iint\limits_R dxdy$$

Example 5. Use double integration to find area of the plane region enclosed by the given curves:

1. $x+y=5,\ x=0,\ y=0,$
2. $y=x^2,\ y=4x,$
3. $y=\sin x,\ y=\cos x$, for $0\le x\le\dfrac{\pi}{4}$,
4. $y^2=-x,\ 3y-x=4,$
5. $y^2=9-x,\ y^2=x-9,$
6. $x+y=2,\ y=x^2,\ y=0.$

Example 6. Sketch the region over which the integration takes place and write an equivalent integral with the order of integration reversed. evaluate both integrals.

$a)\ \int\limits_0^2\int\limits_0^{4-2x} dydx \qquad b)\ \int\limits_0^1\int\limits_2^{4-2x} dydx, \qquad c)\ \int\limits_0^1\int\limits_1^{e^x} dydx, \qquad d)\ \int\limits_0^1\int\limits_0^{4-x^2} dydx,$

$e)\ \int\limits_0^1\int\limits_y^{\sqrt{y}} dxdy \qquad f)\ \int\limits_0^4\int\limits_{\sqrt{y}}^{2} dxdy, \qquad g)\ \int\limits_0^1\int\limits_{-\sqrt{y}}^{\sqrt{y}} dxdy, \qquad h)\ \int\limits_0^1\int\limits_0^{e^y} dxdy.$

From the geometric meaning of double integrals it follows easily that if $f(x,y)$ and $g(x,y)$ are two (continuous) functions defined on a region $R\subset E_2$ and such that for each $[x,y]\in R$ is $f(x,y)\le g(x,y)$, then the volume of the solid S enclosed above by the graph of $f(x,y)$, below by the graph of $g(x,y)$ and laterally by a cylindrical surface generated by vertical lines passing through all points of the boundary of R, is:

$$V(S)=\iint\limits_R (g(x,y)-f(x,y))\,dxdy.$$

Example 7. Find the volume of the solid enclosed by the surface $z=y^2-1$ and by planes $z=3,\ x=0$ and $x=3$.

Example 8. By means of double integration find the volume of the solid in the first octant given by inequalities: $x+y\le z\le 2$.

If a function $f(x,y)$ has continuous first partial derivatives on a closed region $R \subset E_2$, then the area of the surface S given by the equation $z = f(x,y),\ [x,y] \in R$ equals

$$A(S) = \iint_R \sqrt{1 + (f'_x(x,y))^2 + (f'_y(x,y))^2}\ dxdy.$$

Example 9.Find the surface area of the portion

1. of the plane $2x + 2y + z = 8$ in the first octant,
2. of the cylindrical surface $x^2 + z^2 = 4$, above the rectangle $R = \langle 0,1\rangle \times \langle 0,4\rangle \subset E_2$.

Physical Applications.

Let us consider a plate (lamina) in E_2, occupying a regular region R. The plate is supposed to be sufficiently thin so that the mass density is a function of only two variables, x and y, $\sigma(x,y)$. Then the **total mass** M of that plate is:

$$M = \iint_R \sigma(x,y)\ dxdy.$$

By means of double integration we can find also further physical characteristics of the plate, for example **static moments about coordinate axes** and coordinates of its **centre of mass (centre of gravity)**.

Static (first) moments about coordinate axes of the plate are:

$$S_x = \iint_R y.\sigma(x,y)\ dxdy \qquad \text{and} \qquad S_y = \iint_R x.\sigma(x,y)\ dxdy.$$

If we denote by $T = [x_T, y_T] \in E_2$ the centre of mass of the plate, then the coordinates of T are computed as follows:

$$x_T = \frac{S_y}{M} = \frac{\iint_R x.\sigma(x,y)\ dxdy}{\iint_R \sigma(x,y)\ dxdy} \qquad \text{and} \qquad y_T = \frac{S_x}{M} = \frac{\iint_R y.\sigma(x,y)\ dxdy}{\iint_R \sigma(x,y)\ dxdy}.$$

Example 10. A thin plate covers the triangular region bounded by O_x and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density is $\sigma(x,y) = 6x + 6y + 6$. Find the plate's mass, static moments and centre of mass.

Example 11. Find the mass and centre of mass for

1. a rectangular lamina with vertices $[0,0],\ [0,2],\ [3,0],\ [3,2]$, if its density is $\sigma(x,y) = xy^2$,
2. a lamina bounded by O_x, the line $x = 1$ and the curve $y = \sqrt{x}$, with density $\sigma(x,y) = x + y$.

In the special case of a homogeneous region (lamina) R, the centre of gravity is called the **centroid** of the region . In this case

$$x_T = \frac{1}{A(R)} \iint_R x\ dxdy \qquad \text{and} \qquad y_T = \frac{1}{A(R)} \iint_R y\ dxdy$$

Example 12. Find the centroid of the following regions in E_2:

1. The triangular region bounded by lines $y = x,\ x = 1$ and O_x.
2. The region bounded by $y = x^2,\ x = 1$ and O_x.
3. The region enclosed between $y = x$ and $y = 2 - x^2$.
4. The region enclosed between $y = |x|$ and $y = 4$.

The Polar Coordinate System in Plane.

The rectangular coordinate system in plane is not appropriate for all types of problems. There are circumstances in which the **polar coordinate system** is more convenient.

The polar coordinate system is the system of coordinates in plane, in which a point is located by its distance from a fixed point O, called the **pole** or the origin and by the angle that the line from the pole to the given point makes with a fixed half-line (issuing from the pole) called the **polar axis** or polar ray.

The position of a point M in plane is specified by an ordered couple of real numbers $[r, \varphi]$ - the **polar coordinates** of the point M, where $r = |\overline{OM}|$ and φ is the angle formed by the line segment $\overline{OM}$ and the polar axis, issuing from the point O. The angle φ, taken positive when measured counter-clockwise, is called the **polar angle** or amplitude.

It is clear that r is a non negative real number and $r = 0 \implies M \equiv O$. The angle $\varphi \in \langle 0, 2\pi)$. In this way there is one-to-one correspondence between points in plane and their polar coordinates.

Remark 1. Sometimes it is admitted that the angle φ is negative, when measured clockwise, it means that φ ranges from $-\pi$ to π.

When we use both polar and Cartesian coordinates in plane, we usually place the polar origin at the cartesian origin and take the polar axis to be the positive part of O_x. Then the two sets of coordinates are related by the equations:

$$x = r\cos\varphi, \quad y = r\sin\varphi \qquad \text{and} \qquad r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\frac{y}{x}$$

Example 13. Graph the sets of points whose polar coordinates satisfy the following equations and inequalities.

1. $1 \le r \le 2$,
2. $0 \le \varphi \le \pi,\ r = 1$,
3. $\varphi = \dfrac{\pi}{4},\ 2 \le r \le 4$,
4. $\dfrac{\pi}{2} \le \varphi \le \dfrac{3\pi}{4},\ r \le 1$.