8^{th} Lecture

Double Integrals, Basic Properties, Fubini's Theorems.

The Volume of a Curvilinear Cylinder.

Consider the following problem: Let $D \subset E_2$ be a bounded region and f be a nonnegative continuous function of two variables, defined on D. We want to find the volume of a **curvilinear cylinder**, determined by D and f, it means volume of the solid bounded below by D (lying in the plane P_{xy}), above by the surface $z = f(x, y)$ and by the corresponding cylindrical surface, generated by lines, passing through boundary points of D parallel to O_z. The procedure is analogical to that for computing the area of a curvilinear trapezoid.

First we divide D into n subregions $D_1, D_2, \cdots, D_n$, not overlapping and such that areas $A(D_1), A(D_2), \cdots, A(D_n)$ can be computed. Then we choose an arbitrary point from each subregion: $[\xi_i, \eta_i] \in D_i, \quad i = 1, 2, \cdots, n$. Finally we form the sum

$$\sum_{i=1}^{n} f(\xi_i, \eta_i) A(D_i).$$

This number is equal to the volume of a solid, bounded above by parts of planes $z = f(\xi_i, \eta_i)$, therefore depending on the division of the region D and the choice of points $[\xi_i, \eta_i]$. It is natural to consider this sum as an approximation of the desired volume of given curvilinear cylinder.

This idea leads to the concept of double integrals for functions of two variables, over plane regions. In what follows we will discuss a simpler case, if the region D is a two-dimensional interval, it means, a rectangle and the function f is not necessarily nonnegative.

Double Integrals over Intervals.

Let us denote by I two-dimensional interval, which is the cartesian product of two closed intervals, $\langle a, b\rangle$ and $\langle c, d\rangle$.

$$I = \{[x, y] : a \leq x \leq b,\ c \leq y \leq d\} = \langle a, b\rangle \times \langle c, d\rangle$$

Let us take an arbitrary division of the interval $\langle a, b\rangle$:

$$a = x_0 < x_1 < x_2 < \cdots < x_r = b$$

and an arbitrary division of the interval $\langle c, d\rangle$:

$$c = y_0 < y_1 < y_2 < \cdots < y_s = d$$

where r and s are any natural numbers. By these two divisions there is given a division of the interval (rectangle) I, consisting of $n = r.s$ subintervals (rectangles): $I_1, I_2, \cdots, I_n$, such that

$$I = \bigcup_{i=1}^{n} I_i \quad \text{and} \quad A(I) = \sum_{i=1}^{n} A(I_i).$$

Now let f be any function of two variables, defined and bounded on I. In the similar way as above we can compute the sum $\sum\limits_{i=1}^{n} f(\xi_i, \eta_i)A(I_i)$, for any division of I and any choice of points $[\xi_i, \eta_i] \in I_i$. This number is called the **integral sum**.

If there exists limit of integral sums as the area of the greatest subinterval (rectangle) approaches zero, it is called the double integral of f on (over) I and denoted by $\iint\limits_I f(x,y)dxdy$. Therefore:

$$\iint\limits_I f(x,y)dxdy \stackrel{def}{=} \lim_{\max A(I_i) \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)A(I_i)$$

Function f is then called **integrable** on I.

Sufficient condition of integrability.

If a bounded function of two variables possesses only a finite number of points of discontinuity on an interval $I \subset E_2$, then it is integrable on this interval.

Corollary.

Every function of two variables, continuous on an interval $I \subset E_2$ is integrable on I.

Example 1. Show, that the function, defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x.y \text{ is rational,} \\ 0, & \text{if } x.y \text{ is irrational} \end{cases}$$

is not integrable on any two dimensional interval.

Example 2. Show, that the function $f(x,y) = c$, where c is an arbitrary constant is integrable on any two dimensional interval I and $\iint\limits_I f(x,y)dx = cA(I)$.

Basic Properties of Double Integrals over Intervals.

1. **Linearity.** If functions f_1 and f_2 are integrable on an interval I and $c_1, c_2 \in \mathbb{R}$, then

$$\iint\limits_I (c_1 f_1(x,y) + c_2 f_2(x,y))dxdy = c_1 \iint\limits_I f_1(x,y)dxdy + c_2 \iint\limits_I f_2(x,y)dxdy.$$

2. **Additivity.** If a function f is integrable on an interval I and intervals I_1 and I_2 form a division of the interval I $(I_1 \cup I_2 = I,\ A(I_1 \cap I_2) = 0)$, then

$$\iint\limits_I f(x,y)dxdy = \iint\limits_{I_1} f(x,y)dxdy + \iint\limits_{I_2} f(x,y)dxdy.$$

3. **Monotonicity.** If functions f_1 and f_2 are integrable on an interval I and $f_1(x,y) \leq f_2(x,y)$ for each $[x,y] \in I$, then

$$\iint\limits_I f_1(x,y)dxdy \leq \iint\limits_I f_2(x,y)dxdy.$$

Corollary.

If f is a function integrable on I and $f(x,y) \geq 0$ for each $[x,y] \in I$, then

$$\iint_I f(x,y)dxdy \geq 0.$$

Evaluating Double Integrals over Intervals.

Fubini's Theorem (First Form):

If a function of two variables $f(x,y)$ is continuous on an interval $I = \langle a,b\rangle \times \langle c,d\rangle$, then

$$\iint_I f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

Example 3. Compute $\iint_I f(x,y)dxdy$, if

a) $f(x,y) = x^2 + y^2 + 4,\ I = \langle 0,2\rangle \times \langle 0,2\rangle$,

b) $f(x,y) = \sin(2x+y),\ I = \langle 0,\pi\rangle \times \langle \pi/4,\pi\rangle$

Regular Regions in E_2.

The plane region

$$R = \{[x,y] : a \leq x \leq b,\ g(x) \leq y \leq f(x)\} \subset E_2,$$

where $a,b \in \mathbb{R},\ a < b$ and $f(x)$ and $g(x)$ are functions continuous on $\langle a,b\rangle$ and such that for each $x \in \langle a,b\rangle$ it is $g(x) \leq f(x)$, is called a **regular region of the type xy**. The plane region

$$R = \{[x,y] : c \leq y \leq d,\ g(y) \leq x \leq f(y)\} \subset E_2,$$

where $c,d \in \mathbb{R},\ c < d$ and $f(y)$ and $g(y)$ are functions continuous on $\langle c,d\rangle$ and such that for each $y \in \langle c,d\rangle$ it is $g(y) \leq f(y)$, is called a **regular region of the type yx**.

Example 4.Describe the following region $R = \{[x,y] : 1 \leq x \leq 2,\ 1 \leq y \leq x\}$ (of the type xy) as a regular region of the type yx.

Example 5. Describe the following regions as regular regions, or unions of regular regions, for both types, xy and yx.

a) $R = \{[x,y] : 0 \leq x \leq y \leq 1\}$,

b) $R = \{[x,y] : (x^2 + y^2 \geq 1) \wedge (x^2 + 4y^2 \leq 4)\}$

Double Integrals over Regular Regions.

Double integrals of bounded functions of two variables over regular regions are defined similarly as those over two dimensional intervals. It can be proved that analogical properties as those stated for integrals over intervals (linearity, additivity, monotonicity, sufficient condition of integrability) are still valid also for integrals over regular regions.

Evaluating Double Integrals over Regular Regions.

Fubini's Theorem (Strong Form):

If a function of two variables $f(x, y)$ is continuous on a regular region of the type xy

$$R = \{[x, y] : a \leq x \leq b,\ \ g(x) \leq y \leq h(x)\},$$

then

$$\iint\limits_R f(x, y) dx dy = \int\limits_a^b \left(\int\limits_{g(x)}^{h(x)} f(x, y) dy \right) dx.$$

If a function of two variables $f(x, y)$ is continuous on a regular region of the type yx

$$R = \{[x, y] : c \leq y \leq d,\ \ g(y) \leq x \leq h(y)\},$$

then

$$\iint\limits_R f(x, y) dx dy = \int\limits_c^d \left(\int\limits_{g(y)}^{h(y)} f(x, y) dx \right) dy.$$

<u>Example 6.</u> Evaluate $\iint\limits_R \frac{x}{y}\ dxdy$, if $R = \{[x, y] : 1 \leq x \leq y \leq 2\}$.

<u>Example 7.</u> Evaluate $\iint\limits_R ye^x\ dxdy$, if $R = \{[x, y] : y^2 \leq x \leq y + 2\}$.

<u>Example 8.</u> Evaluate $\iint\limits_R (x^2 + y)\ dxdy$, if the region R is bounded by parabolas $y = x^2$ and $x = y^2$.

<u>Example 9.</u> Evaluate $\iint\limits_R (x - y)\ dxdy$, if the region R is bounded by straight lines $y = 0,\ \ y = x,\ \ x + y = 2$.

<u>Example 10.</u> Show, that we are able to evaluate $\iint\limits_R e^{y^2}\ dxdy$, where R is the region from the Example 5.a) above, only if R is described as a regular region of the type yx.