# Linear Second Order Non-homogeneous Differential Equations with Constant Coefficients

Consider the equation

$$(1) \qquad y'' + p_1 y' + p_2 y = g(x)$$

$p_1, p_2 \in R$, $g(x)$ is a function continuous on an interval $(a,b)$.

The equation

$$(2) \qquad y'' + p_1 y' + p_2 y = 0$$

(with the same $p_1$ and $p_2$) is said to be the **homogeneous equation, associated with (1)**.
If $Y = c_1 y_1 + c_2 y_2$ is a general solution of the equation (2) on $R$ and $y_p$ is arbitrary solution of (1) on $(a,b)$, then

$$y = Y + y_p = c_1 y_1 + c_2 y_2 + y_p$$

is the general solution of (1) on $(a,b)$.

Since any general solution of (1) is sum of a particular solution of (1) and general solution of (2), the equation (1) is solved as follows:

1) We find general solution of the associated equation (2), $Y$
2) We find one particular solution of (1), $y_p$
3) Then the general solution of (1) is: $y = Y + y_p$

Example 1. Find general solution of the equation $y'' + y = 2x$, if you know that one particular solution is $y_p = 2x$.

There are two methods, how to find a particular solution of an equation (1).

**I. The method of variation of constants.**

Let $Y = c_1 y_1 + c_2 y_2$ be general solution of the equation (2), $c_1, c_2 \in R$. We try to find a particular solution of (1) on $(a,b)$ in the form

$$y_p = c_1(x) y_1 + c_2(x) y_2,$$

it means that constants $c_1$ and $c_2$ we replace by functions $c_1(x)$ and $c_2(x)$. We look for such functions $c_1(x)$ and $c_2(x)$, that $y_p$ satisfies the equation (1). By means of Cramer's Rule it can be proved, that it is fulfilled, if $c_1'(x) = \dfrac{W_1(x)}{W(x)}$ and $c_2'(x) = \dfrac{W_2(x)}{W(x)}$, where $W_i(x)$ is determinant obtained from the $W(x) = W(y_1, y_2)$ by replacing the i-th column by the column $\begin{pmatrix} 0 \\ g(x) \end{pmatrix}$, $i = 1, 2$.

Therefore $c_1(x) = \displaystyle\int \frac{W_1(x)}{W(x)} dx$, $c_2(x) = \displaystyle\int \frac{W_2(x)}{W(x)} dx$ and hence

$$y_p = y_1\int\frac{W_1(x)}{W(x)}dx + y_2\int\frac{W_2(x)}{W(x)}dx$$

<u>Example 2</u>. Solve the equation $y'' - y' = x + 1$

<u>Example 3</u>. Solve the equation $y'' + y = \frac{1}{\cos x}$, on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

<u>Example 4.</u> Find a solution of the equation $y'' + 6y' + 9 = \frac{e^{-3x}}{x^2}$; satisfying initial conditions: $y(1) = 0$, $y'(1) = 0$.

## II. The method of undetermined coefficients.

Solving non-homogeneous equations with a **special form of right-hand member**.
If the right-hand member $g(x)$ of an equation (1) has a form $g(x) = e^{\alpha x}\cdot P(x)$, where $\alpha \in R$ and $P(x)$ is an m-th degree polynomial, the there exists a particular solution of the equation (1):

$$y_p = x^k \cdot e^{\alpha x}\cdot P(x)$$

where $k$ is the multiplicity of $\alpha$ considered as a root of the characteristic equation and $P(x) = b_0 + b_1 x + ... + b_m x^m$ is an unknown polynomial of the same degree as $P(x)$. Coefficients $b_0, b_1, ... b_m$ are found by the method of undetermined coefficients.

<u>Example 5.</u> Find one particular solution of the equation $y'' - y' = x + 1$

<u>Example 6.</u> Solve the equation $y'' - 2y' + y = e^x$

If the right-hand member of the equation (1) has a form $g(x) = e^{\alpha x}(P(x)\cos\beta x + Q(x)\sin\beta x)$, where $\alpha, \beta \in R$ and $P(x)$ and $Q(x)$ are polynomials, then there exists a particular solution of the equation (1).

$$y_p = x^k \cdot e^{\alpha x}(P^*(x)\cos\beta x + Q^*(x)\sin\beta x),$$

where $k$ is multiplicity of $\alpha + i\beta$ considered as a root of the characteristic equation and $P^*(x)$ and $Q^*(x)$ are unknown polynomials of the same degree identical with the greater of degrees of polynomials $P(x)$ and $Q(x)$. Coefficients of $P^*(x)$ and $Q^*(x)$ are found by the method of undetermined coefficients.

Example 7. Solve the equation $y'' + 2y' + 5y = 2\cos x$

Example 8. Solve the equation $y'' + 4y = \sin 2x$

Advantage of the method of undetermined coefficients is that this method doesn't require integration and its application (in the case of special form of right-hand member) is for the most part much more simpler than the variation of constants.

Disadvantage of this method is that it is restricted to the case of special form of right-hand member, hence it is not possible to use it always, contrary to the method of variation of constants, which is general.

Example 9. Find particular solution of the equation $y'' - 2y' + y = \frac{e^x}{x}$, satisfying the initial conditions $y(1) = 1$, $y'(1) = 0$.

Example 10. Find particular solution of the equation $y'' - 2y' + 2y = \frac{e^x}{\sin x}$, satisfying the initial conditions $y\left(\frac{\pi}{2}\right) = 0$, $y'\left(\frac{\pi}{2}\right) = 1$

Example 11. Find particular solution of the equation $y'' - 2y' = e^{2x}$, satisfying the initial conditions $y(0) = 2$, $y'(0) = \frac{1}{2}$.

Example 12. Find particular solution of the equation $y'' - 6y' + 9y = 3\cos 3x$, satisfying the initial conditions $y(0) = 1$, $y'(0) = \frac{1}{2}$.