11^{th} Lecture

Triple Integrals, Basic Properties, Fubini's Theorems.

The Mass of a Nonhomogeneous Body.

Consider the following problem: Let a physical body occupy a bounded region D, in space $D \subset E_3$ and let this body be nonhomogeneous, it means, its mass density varies, it depends on the position of a point in the body. We can assume, that the density is represented by a nonnegative continuous function of three variables $\sigma(x, y, z)$, defined on D. We want to find the total mass of this nonhomogeneous solid. The procedure how to estimate this mass is analogical to that, for computing the volume of a curvilinear cylinder.

We again divide D into n subregions $D_1, D_2, \cdots, D_n$, not overlapping and such, that volumes $V(D_1), V(D_2), \cdots, V(D_n)$ can be computed. Then we choose an arbitrary point from each subregion: $[\xi_i, \eta_i, \tau_i] \in D_i, \quad i = 1, 2, \cdots, n$. Finally we form the sum

$$\sum_{i=1}^{n} \sigma(\xi_i, \eta_i, \tau_i) V(D_i).$$

This number is equal to the mass of a "by parts homogeneous" solid, depending on the division of the region D and the choice of points $[\xi_i, \eta_i, \tau_i]$. It is natural to consider this sum as an approximation of the desired mass of given physical body.

This idea leads to the concept of triple integrals for functions of three variables, over space regions. In what follows we will discuss a simpler case, if the region D is a three-dimensional interval, it means, a **rectangular parallelepiped** and the function $f(x, y, z)$ $(\sigma(x, y, z))$ is not necessarily nonnegative.

Triple Integrals over Intervals.

Let us denote by I a three-dimensional interval, which is the Cartesian product of three closed intervals, $\langle a, b\rangle$, $\langle c, d\rangle$ and $\langle e, f\rangle$.

$$I = \{[x, y, z] : a \le x \le b, \ c \le y \le d, \ e \le z \le f\} = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, f\rangle$$

Let us take an arbitrary division of the interval $\langle a, b\rangle$:

$$a = x_0 < x_1 < x_2 < \cdots < x_r = b,$$

an arbitrary division of the interval $\langle c, d\rangle$:

$$c = y_0 < y_1 < y_2 < \cdots < y_s = d$$

and an arbitrary division of the interval $\langle e, f\rangle$:

$$e = z_0 < z_1 < z_2 < \cdots < z_t = d,$$

where r, s and t are any natural numbers. By these three divisions there is given a division of the interval (parallelepiped) I, consisting of $n = r.s.t$ subintervals (rectangular parallelepipeds) $I_1, I_2, \cdots, I_n$, such that

$$I = \bigcup_{i=1}^{n} I_i \quad \text{and} \quad V(I) = \sum_{i=1}^{n} V(I_i).$$

Now let f be any function of three variables, defined and bounded on I. In the similar way as above we can compute the sum $\sum\limits_{i=1}^{n} f(\xi_i, \eta_i, \tau_i)V(I_i)$, for any division of I and any choice of points $[\xi_i, \eta_i, \tau_i] \in I_i$. This number is called the **integral sum**.

If there exists limit of integral sums as the volume of the greatest subinterval (parallelepiped) approaches zero, it is called the triple integral of f on (over) I and denoted by $\iiint\limits_I f(x, y, z)dxdydz$. Therefore:

$$\iiint\limits_I f(x, y, z)dxdydz \stackrel{def}{=} \lim_{\max V(I_i)\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \tau_i)V(I_i)$$

Function f is then called **integrable** on I.

Sufficient condition of integrability.

If a bounded function of three variables possesses only a finite number of points of discontinuity on an interval $I \subset E_3$, then it is integrable on this interval.

Corollary.

Every function of three variables, continuous on an interval $I \subset E_3$ is integrable on the interval I.

Example 1. Show, that the function $f(x, y, z) = c$, where c is an arbitrary constant is integrable on any three dimensional interval I and $\iiint\limits_I f(x, y, z)dxdydz = c.V(I)$.

Basic Properties of Triple Integrals over Intervals.

1. **Linearity.** If functions f_1 and f_2 are integrable on an interval I and $c_1, c_2 \in \mathbb{R}$, then

$$\iiint\limits_I (c_1 f_1(x, y, z)+c_2 f_2(x, y, z))dxdydz = c_1 \iiint\limits_I f_1(x, y, z)dxdydz+c_2 \iiint\limits_I f_2(x, y, z)dxdydz.$$

2. **Additivity.** If a function f is integrable on an interval I and intervals I_1 and I_2 form a division of the interval I $(I_1 \cup I_2 = I,\ V(I_1 \cap I_2) = 0)$, then

$$\iiint\limits_I f(x, y, z)dxdydz = \iiint\limits_{I_1} f(x, y, z)dxdydz + \iiint\limits_{I_2} f(x, y, z)dxdydz.$$

3. **Monotonicity.** If functions f_1 and f_2 are integrable on an interval I and $f_1(x,y,z) \leq f_2(x,y,z)$ for each $[x,y,z] \in I$, then

$$\iiint_I f_1(x,y,z)dxdydz \leq \iiint_I f_2(x,y,z)dxdydz.$$

Evaluating Triple Integrals over Intervals.

Fubini's Theorem (First Form):

If a function of three variables $f(x,y,z)$ is continuous on an interval $I = \langle a,b\rangle \times \langle c,d\rangle \times \langle e,f\rangle$, then

$$\iiint_I f(x,y,z)dxdydz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)dx \right) dy \right) dz.$$

In computing triple integrals over intervals by means of this theorem we can interchange order of repeated integrals on the right-hand side of the above equality. In fact, we have 6 possibilities, how to rearrange them.

<u>Example 2.</u> By means of Fubini's Theorem compute $\iiint_I (x^3 + xy + yz)dxdydz$, if $I = \langle 0,1\rangle \times \langle 0,2\rangle \times \langle 1,3\rangle$.

Regular Regions in E_3.

In E_3 we distinguish 6 types of regular region: xyz, yxz, zxy, xzy, yzx, zyx. For example, a set

$$R = \{[x,y,z] : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ \varphi(x,y) \leq z \leq \psi(x,y)\} \subset E_3,$$

where $a, b \in \mathbb{R}$ and $f(x)$, $h(x)$, $\varphi(x,y)$ and $\psi(x,y)$ are continuous functions of one or two variables, is called a **regular region of the type xyz**. The other types of regular regions are defined analogously.

<u>Example 3.</u>Describe the set $R = \{[x,y,z] : x^2 + y^2 \leq z^2 \leq 4,\ z \geq 0\}$ as a regular region of the type xyz.

Triple Integrals over Regular Regions.

Triple integrals of bounded functions of three variables over regular regions are defined similarly as those over three dimensional intervals. It can be proved that analogical properties as those stated for integrals over intervals (linearity, additivity, monotonicity, sufficient condition of integrability) are still valid also for integrals over regular regions.

Evaluating Triple Integrals over Regular Regions.

Fubini's Theorem (Strong Form):

If a function of three variables $f(x,y,z)$ is continuous on a regular region of the type xyz

$$R = \{[x,y,z] : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ \varphi(x,y) \leq z \leq \psi(x,y)\},$$

then

$$\iiint\limits_R f(x,y,z)dxdydz = \int\limits_a^b \left(\int\limits_{g(x)}^{h(x)} \left(\int\limits_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \right) dy \right) dx.$$

Analogical propositions are valid also for the other types of regular regions.

Example 4. Evaluate $\iiint\limits_R (x+y+z)\ dxdydz$, if
$R = \{[x,y,z] : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$.

Basic Applications of Triple Integrals.

If $R \subset E_3$ is a regular region (of any type), then its volume $V(R)$ is computed by

$$V(R) = \iiint\limits_R dxdydz.$$

Example 5. Find the volume of the solid S, if S is a region given by inequalities

a) $3x^2 + 3y^2 \leq z \leq 1 - x^2 - y^2$,

b) $x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y \leq 2,\ z \leq 1 - x^2$.

If a physical body occupies a regular region $R \subset E_3$ and its point density is a function $\sigma(x,y,z)$, then the total mass of the body is

$$M = \iiint\limits_R \sigma(x,y,z)dxdydz.$$

If we denote by $T = [x_T, y_T, z_T] \in E_3$ the centre of mass of the body, then the coordinates of T are computed as follows:

$$x_T = \frac{1}{M} \iiint\limits_R x.\sigma(x,y,z)\ dxdydz, \qquad y_T = \frac{1}{M} \iiint\limits_R y.\sigma(x,y,z)\ dxdydz,$$

$$z_T = \frac{1}{M} \iiint\limits_R z.\sigma(x,y,z)\ dxdydz.$$

Example 6. Compute the total mass of the solid S

a) bounded by surfaces $z = \sqrt{x^2+y^2},\ 2z = x^2+y^2$, if $\sigma(x,y,z) = z^2$,

b) given by inequalities $\sqrt{x^2+y^2} \leq z \leq 2,\ x \geq 0,\ y \geq 0$, if $\sigma(x,y,z) = \dfrac{xy^3z}{(1+z^2)^2}$.

Example 7. Find coordinates of the centre of mass of a homogeneous tetrahedron, bounded by planes $x=0,\ y=0,\ z=0$ and $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ $(a>0,\ b>0,\ c>0)$.