Differential Equations of the 1st order

Basic Notions

Many physical, chemical or technical problems lead to differential equations.
An ordinary differential equation is an equation which involves one independent variable x, an unknown function $y = f(x)$ and its derivatives $y', y'', ... y^{(n)}$. In general a differential equation can be written as follows $F\left(x, y, y', ... y^{(n)}\right) = 0$. The order of a differential equation is the order of the highest derivative which appears.

Every function which, when substituted, together with its derivatives into the given differential equation, turns it into identity on a set *M* is called a solution (or an integral) of the differential equation on the set *M*.

Differential Equations of the 1st order

The general form of the 1st order differential equation is $F(x, y, y') = 0$. There exist 1st order differential equations, having no solution, for example: $(y')^2 + x^2 + y^2 + 1 = 0$ But in general case, a 1st order differential equation has infinitely many solutions, expressed by a formula $y = \varphi(x, c)$**,** containing an arbitrary constant c. Such family of solutions is called the **general solution**. The general solution is not always expressible in an explicit form and sometimes we represent it in an **implicit form** $\phi(x, y, c) = 0$.

A **particular solution** is any function $y = \varphi(x, c_\circ)$, which is obtained from the general solution, when we assign to the arbitrary constant a definite value $c = c_\circ$. In what follows when solving concrete equations we'll most often be concerned with particular solutions specified by the **initial condition** (Cauchy's initial condition): $y(x_\circ) = y_\circ$
A solution, not obtained from the general solution and not containing any constant is called a **singular solution**.

<u>Example 1.</u> Consider the equation: $y' y - ye^x = 0$. Verify, that $y = e^x + 1$ is the particular solution, satisfying the initial condition: $y(0) = 2$. The function $y = 0$ is the singular solution.

Graph of a solution is called the **integral curve** of the given differential equation.

<u>Example 2.</u> Cooling of a body: According to the law established by Newton, the rate of cooling of a physical body is directly proportional to the difference between the temperature of the body and that of surrounding medium. Let at the time $t = t_\circ = 0$ the temperature of the body be $T_\circ > 0$ $\left(T(0) = T_\circ\right)$. We want to determine the relationship between the variable temperature of body T and the time t. Let's suppose, that the temperature of the medium is 0. By Newton's law: $\frac{dT}{dt} = -k(T - 0) = -kT$, where k is the proportionality factor. It can be shown, that each function $T = Ce^{-kt}$ is the particular solution satisfying the given initial condition.

Differential Equations with Separated Variables

Differential equations $p(x)+q(x)y'=0$ (1) where $p(x)$ is a function continuous on an interval (a,b) and $q(y)$ on an interval (c,d) are called 1st order differential equations with separated variables.

Each solution of the equation (1) on an interval $J\subset(a,b)$ has the form: $\int p(x)dx+\int q(y)dy=C$, what is the general solution in implicit form.

Remark. If $q(y)\neq 0$ on (c,d), then through each point form the region $D=(a,b)\times(c,d)\subset E_2$ is passing just one integral curve of the equation (1).

Example 3. a) Solve the equation $2x+\frac{y'}{y}=0$

b) Find the particular solution of the equation $x+yy'=0$, satisfying the initial condition $y(3)=4$

A special case of the differential equation (1) are equations of the form $y'=f(x)$, with t e general solution $y=\int f(x)dx+C$

Example 4. a) Find the particular solution of the equation $y'=3x^2$, satisfying $y(1)=2$

b) Solve the equation $y'=\frac{1}{2\sqrt{x}}$

Differential Equations with Separable Variables

Equations of the form $p_1(x)p_2(y)+q_1(x)q_2(y)y'=0$ (2) are called 1st order differential equations with separable variables, $p_1(x)$ and $q_1(x)$ are supposed to be continuous on (a,b), $p_2(y)$ and $q_2(y)$ on (c,d).

Under the condition $q_1(x)\cdot p_2(x)\neq 0$, the equation (2) can be reduced to $\frac{p_1(x)}{q_1(x)}+\frac{q_2(x)}{p_2(x)}y'=0$ (3) .

Equations (2) and (3) are not completely equivalent. If $p_2(y)=0$, for $y_1=b_1$, $y_2=b_2$, … $y_k=b_k$, where $b_i\in(c,d)$ $i=1,2,\ldots k$ then functions y = b_i are solutions of the equation (2).

It follows, that solution of the equation (2) are all function $y=b_i$ and all solutions of the equation with separated variables (3), it means of the form

$$\int \frac{p_1(x)}{q_1(x)}dx + \int \frac{q_2(y)}{p_2(y)}dy = C, \quad C \in R$$

Example 5. Solve the equations: a) $y - xy' = 0$, b) $\frac{y^2+4}{x} + yy' = 0$

Example 6. Find the particular solution of the equation $y' = \frac{2xy}{1+x^2}$, satisfying the initial condition $y(1) = -1$

Linear Differential Equations of the 1st order

Differential equations $y' + p(x)y = q(x)$ (4) where $p(x)$ and $q(x)$ are continuous on (a,b) are called **non-homogeneous** (with right hand member) linear differential equation, if $q(x)$ is a nonzero function. If $q(x) = 0$ on (a,b), it means: $y' + p(x)y = 0$ (5) is called **homogeneous** (without right hand member) linear differential equation.

The equation (5) is separable and it can be easily shown, that $y = Ce^{-\int p(x)dx}$, where C is a constant, is the general solution of (5) on (a,b).

A non-homogeneous linear dif. equation (4) is solved by the **method of variation of a constant**. First we find the general solution o the associated linear differential equation (5) and then we look for a solution of (4) in the form $y = C(x)e^{-\int p(x)dx}$, where $C(x)$ is such a function that y satisfies the equation (4). Thus $C(x) = \int g(x)e^{\int p(x)dx} + C$ and consequently

$$y = \left[\int g(x)e^{\int p(x)dx} + C\right]e^{-\int p(x)dx} = Ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int g(x)e^{\int p(x)dx} \quad , \quad C \in R$$

The general solution of the equation (4) is always expressible as a sum of the general solution of (5) and one particular solution of (4).

Example 7. Solve equations:

a) $y' - \frac{y}{x} = x^2$, b) $y' - y\cot x = 2x\sin x$, $y\left(\frac{\pi}{2}\right) = 0$

c) $y' - \frac{1}{x}y = \frac{\sin x}{x}$, $y(\pi) = 0$, d) $y' - \frac{2}{x+1}y = (x+1)^3$, $y(0) = \frac{3}{2}$