INTERPOLATION

Polynomial $P_n(x)$ interpolates the data points $[x_i, y_i]$, $i = 1, 2, ..., k$ if and only if $P_n(x_i) = y_i$, for each $i = 1, 2, \dots, k$. The degree n of interpolating polynomial $P_n(x)$ is less or equal $k - 1$.

Exercise 1:

With respect to the definition find the interpolating polynomial fitting the points:

i	1	2	3
x_i	-1	0	2
y_i	2	1	5

Solution:

There are three points given, the degree of interpolating polynomial is no more than $n = 2$. The form of required polynomial will be: $P_2(x) = a\,x^2 + b\,x + c$. For all data points: $P_n(x_i) = y_i$. The system of three linear eqations is solved :

$i = 1$: $a.(-1)^2 + b\,(-1) + c = 2$

$i = 2$: $a.0 + b.0 + c = 1$

$i = 3$: $a.4 + b.2 + c = 5$

```
Solve[{a-b+c==2,c==1,4a+2b+c==5},{a,b,c}]
```

(a, b, c) = (1, 0, 1).

The required polynomial $P_2(x) = x^2 + 1$.

Newton's interpolating polynomial

has a form:

$$N_n(x) = a_0 + a_1(x-x_1) + a_2(x-x_1)(x-x_2) + \dots\dots + a_n(x-x_1)(x-x_2)\dots\dots(x-x_n)$$

where $a_0 = f(x_1)$, $a_1 = \Delta f(x_1) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$, $a_2 = \Delta^2 f(x_1) = \dfrac{\Delta f(x_2) - \Delta f(x_1)}{x_3 - x_1}$,....

$$a_i = \Delta^i f(x_1) = \frac{\Delta^{i-1} f(x_2) - \Delta^{i-1} f(x_1)}{x_{i+1} - x_1}.$$

$\Delta^i f(x) = \Delta(\Delta^{i-1} f(x))$ is the i-th devided difference of function f in x.

The degree of the polynomial is n, and the number of points is n+1.

Exercise 2:

Find the interpolating polynomial for the following data: {[-0.1; 4.5], [0; 4], [0.5; 7.2], [1.2; 10.4]}. Use divided difference table.

Solution:

Divided difference table:

i	x_i	$y_i = f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$
1	-0.1	4.5			
			-5		
2	0	4		19	
			6.4		-15.7875
3	0.5	7.2		-1.52381	
			4.57143		
4	1.2	10.4			

Put the values into formula for $N_n(x)$. There are four points in the set. The polynomial will be of degree no more than three:

$$N_3(x) = a_0 + a_1(x-x_1) + a_2(x-x_1)(x-x_2) + a_3(x-x_1)(x-x_2)(x-x_3)$$

$$= 4{,}5 + (-5)(x+0{,}1) + 19(x+0.1)\,x - 15.7875(x+0.1)\,x\,(x-0.5)$$

$$= 4 - 2.31062x + 25.315x^2 - 15.7875x^3.$$

In *Mathematica* one can use the command:

```
InterpolatingPolynomial[M,x]//Expand
```

Interpolating polynomial is unambiguously appointed by data points. It does not depend on calculation method.

Exercise 3:

Find Newton's interpolating polynomial of degree 3 in the way to approximate the specific value for **x = 4,3** the most precisely.

x	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5
$f(x)$	0	2.3	4.2	2.7	3.2	3.7	3.0	4.3	4.5	4.7	3.9	4.1

Solution:

To specificate the most exact value *f*(4.3) by the polynomial of the third degree, 4 suitable points have to be chosen in the vicinity of x = 4.3.

2.5	3.5	4.5	5.5
2.3	4.2	2.7	3.2

Using *Mathematica* we obtain

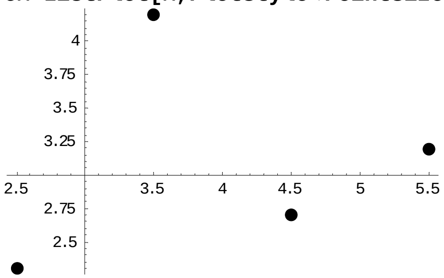
```
M={{2.5,2.3},{3.5,4.2},{4.5,2.7},{5.5,3.2}}
 {{2.5,2.3},{3.5,4.2},{4.5,2.7},{5.5,3.2}}

 P3[x_]=InterpolatingPolynomial[M,x]//Expand
```

$-52.7625 + 44.275\,x - 11.15\,x^2 + 0.9\,x^3$

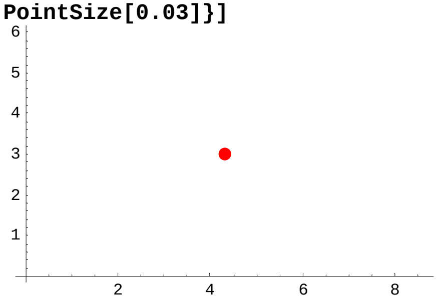
```
 P3[4.3]
 3.0128
```

We draw the plots:

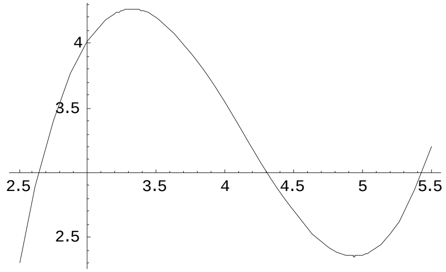
```
GM=ListPlot[M,PlotStyle→PointSize[0.03]]
```



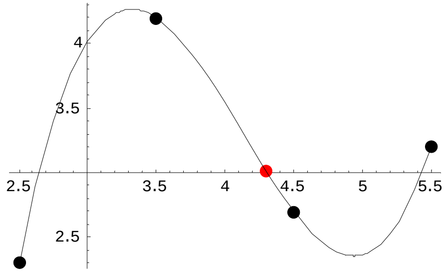

```
-Graphics-
 GB=ListPlot[{{4.3,P3[4.3]}},PlotStyle→{RGBColor[1, 0, 0],
PointSize[0.03]}]
```




```
-Graphics-
  GP=Plot[P3[x],{x,2.5,5.5}]
```


```
-Graphics-
Show[GM,GB,GP]
```




```
-Graphics-
```

Note. *Take notice of the graph of P3(x) drawn above all data points as well as of the interpolating polynomial and it's graph createtd over all data points.*

Exercise. 4:

a) Create the set of data points $\{[x, e^x]\}$ for $x \in \langle 0;3 \rangle$ with step 0.5.

b) Find out the corresponding interpolating polynomial.

c) Calculate the interpolation error for $x = 2.3$.

d) Draw the plots of the functions e^x and interpolating polynomial on interval $\langle -3;6 \rangle$.
Compare the graphs and behavior of the functions inside and outside the interval.

Solution:

Creation of set of points:

M=Table[{x,Exp[x]},{x,0,3,0.5}] $\Rightarrow$ {{0, 1}, {0.5, 1.64872}, {1., 2.71828}, {1.5, 4.48169}, {2., 7.38906}, {2.5, 12.1825}, {3., 20.0855}}

Interpolating polynomial:

```
P[x_]=InterpolatingPolynomial[M,x]//Expand
```
$\Rightarrow$

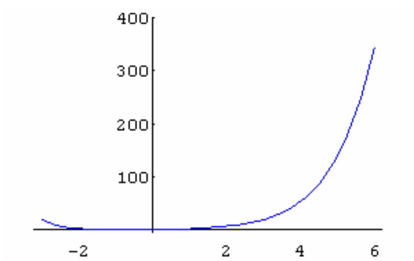
$1 + 0.991168\, x + 0.542326\, x^2 + 0.0915477\, x^3 + 0.105665\, x^4 - 0.0190508\, x^5 + 0.00662516\, x^6$

The interpolation error for $x = 2.3$:

```
Delta = |Exp[2.3] - P[2.3] |
```
$\Rightarrow$ 0.00018713

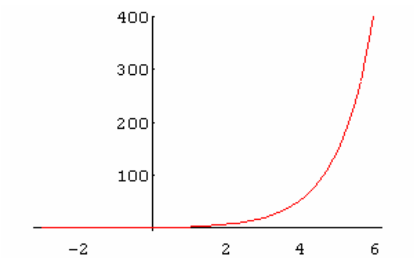
Plot of the interpolating polynomial (blue):

```
GP=Plot[P[x],{x,-3,6},PlotRange->All,PlotStyle->RGBColor[0,0,1]]
```

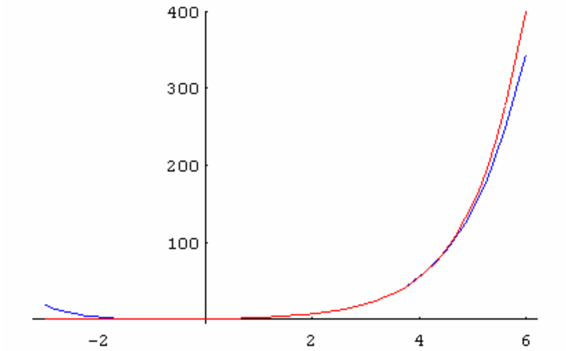


Plot of the function e^x (red):

```
GF=Plot[Exp[x],{x,-3,6},PlotRange->All,PlotStyle->RGBColor[1,0,0]]
```




```
Show[GP,GF]
```

Notice that the distance between graphs out of interval $\langle 0,3 \rangle$ is growing.