Diferenciálne rovnice
- Numerické jednokrokové metódy

Úloha :

Riešte diferenciálnu rovnicu
$y + x\,y'\ln x = 0$, $y(e) = 3$,
pre $x \in \langle e, 2+e \rangle$ a krok $h=0{,}1$.

Čiastkové úlohy:
a) Vyšetrite existenciu a jednoznačnosť riešenia.
b) Ak je to možné, určte presné všeobecné a partikulárne riešenie. Použite príkaz DSolve.
c) Eulerovou metódou nájdite približné riešenie.
d) Nájdite približné riešenie metódou Runge-Kutta 2. rádu.
e) Nájdite približné riešenie metódou Runge-Kutta 3. rádu.
f) Nájdite približné riešenie metódou Runge-Kutta 4. rádu.
g) V koncovom bode intervalu x = 2+ e vypočítajte globálne chyby všetkých približných riešení. Porovnajte.
h) Na určenie numerického riešenia použite príkaz NDSolve.

Riešenie:

a) Vyšetrite existenciu a jednoznačnosť riešenia.

Rovnicu riešime v tvare y' = $f(x,y)$

```
f[x_, y_] = -y / (x * Log[x])
```

$$-\frac{y}{x\,\text{Log}[x]}$$

Ak je funkcia $f(x,y)$ v začiatočnom bode $y(a)$=y_a spojitá, potom existuje riešenie úlohy.
Funkcia $-\frac{y}{x\,\ln(x)}$ je definovaná pre všetky $x \in R^+$. Pre $x = e$ je funkcia definovaná a spojitá.
Riešenie úlohy existuje.

Ak je parciálna derivácia $\frac{\partial f(x,y)}{\partial y}$ v začiatočnom bode $y(a)$=y_a spojitá, , potom riešenie je jenoznačné.
$\frac{\partial f(x,y)}{\partial y} = \frac{-1}{x\,\ln(x)}$ je definovaná pre všetky $x \in R^+$. Pre $x = e$ je parciálna derivácia definovaná
a spojitá.
Riešenie je jednoznačné.

b) Ak je to možné, určte presné všeobecné a partikulárne riešenie. Použite príkaz DSolve.

Všeobecné riešenie:

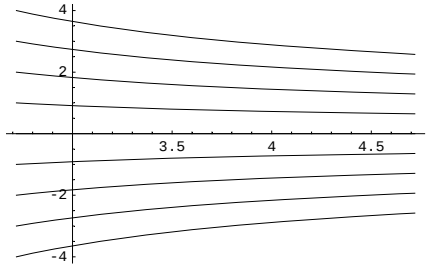
```
Clear[ygesol, x, y];
DSolve[y[x] + x * y'[x] * Log[x] == 0, y[x], x]
```

$$\{\{y[x] \to \frac{C[1]}{\text{Log}[x]}\}\}$$


```
ygesol[x_] = c / Log[x]
```

$$\frac{c}{\text{Log}[x]}$$

```
GGS = Table[Plot[ygesol[x], {x, E, 2 + E}], {c, -4, 4}];
Show[GGS]
```



- Graphics -

Partikulárne riešenie:

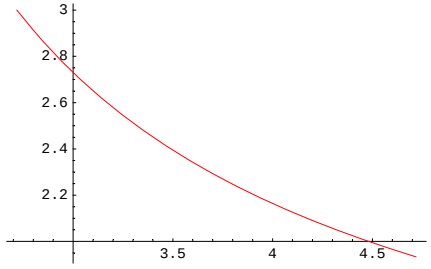
```
Clear[t, y, rie]
DSolve[{y[x] + x * y'[x] * Log[x] == 0, y[E] == 3}, y[x], x]
```

$$\{\{y[x] \to \frac{3}{\text{Log}[x]}\}\}$$

```
yps[x_] = 3 / Log[x]
```

$$\frac{3}{\text{Log}[x]}$$

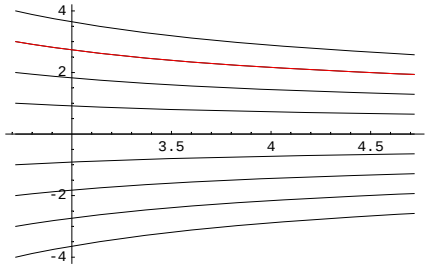
```
GPS = Plot[yps[x], {x, E, 2 + E}, PlotStyle → RGBColor[1, 0, 0]]
```



- Graphics -

Graf partikulárneho a všeobecného riešenia

```
Show[GGS, GPS]
```




```
- Graphics -
```

Numerické jednokrokové metódy

c) Eulerova metóda

```
h = 0.1; a = E; b = 2 + E;
iter = (b - a) / h
```

```
20.
```

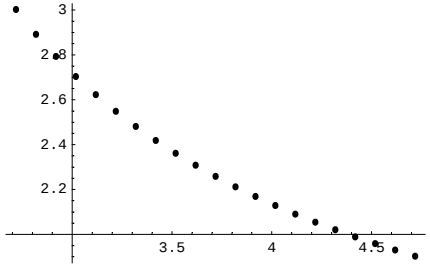
```
x[0] = E;
yE[0] = 3;
Do[
 x[i + 1] = x[i] + h;
 yE[i + 1] = yE[i] + h * f[x[i], yE[i]],
 {i, 0, iter}]
```

```
EU = Table[{x[i], yE[i]}, {i, 0, iter}];
TableForm[EU, TableHeadings → {None, {"x", "yEuler"}}]
```

x	yEuler
e	3
2.81828	2.88964
2.91828	2.79068
3.01828	2.70139
3.11828	2.62037
3.21828	2.54648
3.31828	2.47879
3.41828	2.41651
3.51828	2.35899
3.61828	2.30569
3.71828	2.25614
3.81828	2.20994
3.91828	2.16674
4.01828	2.12625
4.11828	2.0882
4.21828	2.05238
4.31828	2.01858
4.41828	1.98662
4.51828	1.95636
4.61828	1.92765
4.71828	1.90037

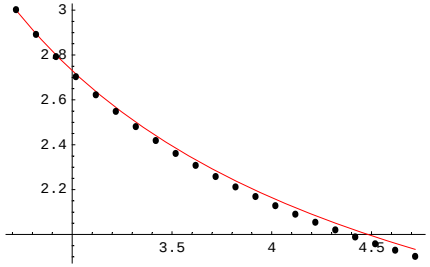
Graf približného riešenia Eulerovou metódou

```
GE = ListPlot[EU, PlotStyle → PointSize[0.015]]
```




```
- Graphics -
```

```
Show[GPS, GE]
```




```
- Graphics -
```

absolútna chyba riešenia v koncovom bode intervalu x =2+e

```
x[20]
```

```
4.71828
```

```
EpsEU = Abs[yps[x[20]] - yE[20]]
```

```
0.0333115
```

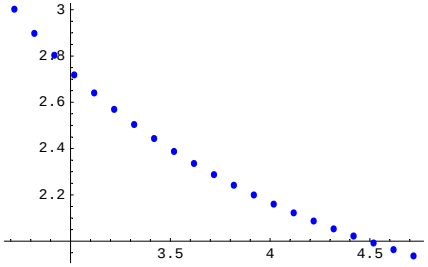
d) Metóda RUNGE - KUTTA 2. rádu

```
Clear[yRK2]
x[0] = E;
yRK2[0] = 3;
Do[
    k1 = f[x[i], yRK2[i]];
    k2 = f[x[i] + h / 2, yRK2[i] + h / 2 * k1];
   yRK2[i + 1] = yRK2[i] + h * k2;
  x[i + 1] = x[i] + h,
   {i, 0, iter}];
```

```
RK2 = Table[{x[i], yRK2[i]}, {i, 0, iter}];
TableForm[RK2, TableHeadings → {None, {"x", "yRunge-Kutta 2"}}]
```

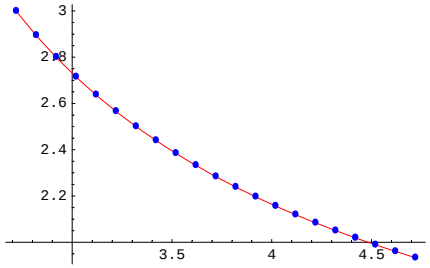
x	yRunge-Kutta 2
e	3
2.81828	2.89553
2.91828	2.80136
3.01828	2.71601
3.11828	2.63823
3.21828	2.56704
3.31828	2.5016
3.41828	2.44121
3.51828	2.38529
3.61828	2.33334
3.71828	2.28492
3.81828	2.23968
3.91828	2.1973
4.01828	2.1575
4.11828	2.12005
4.21828	2.08473
4.31828	2.05135
4.41828	2.01975
4.51828	1.98978
4.61828	1.96132
4.71828	1.93425

```
GRK2 = ListPlot[RK2, PlotStyle → {PointSize[0.015], RGBColor[0, 0, 1]}]
```



- Graphics -


```
Show[GPS, GRK2]
```



- Graphics -

absolútna chyba riešenia v koncovom bode intervalu x =2+e

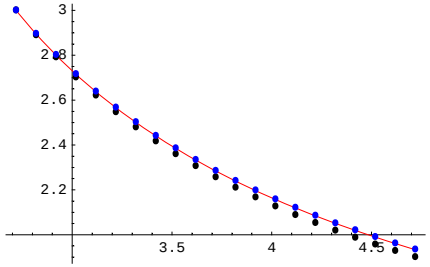
```
x[20]
```

4.71828

```
EpsRK2 = Abs[yps[x[20]] - yRK2[20]]
```

0.00056469

```
Show[GPS, GE, GRK2]
```



- Graphics -

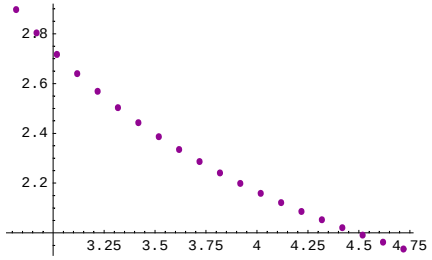
e) RUNGE - KUTTA ORDER 3

```
x[0] = E;
yRK3[0] = 3;
Do[
    k1 = f[x[i], yRK3[i]];
    k2 = f[x[i] + h / 3, yRK3[i] + h / 3 * k1];
    k3 = f[x[i] + h * 2 / 3, yRK3[i] + h * 2 / 3 * k2];
   yRK3[i + 1] = yRK3[i] + h / 4 * (k1 + 3 * k3);
  x[i + 1] = x[i] + h,
   {i, 0, iter}];
```

```
RK3 = Table[{x[i], yRK3[i]}, {i, 1, iter}];
TableForm[RK3, TableHeadings → {None, {"x", "yRunge-Kutta 3"}}]
```

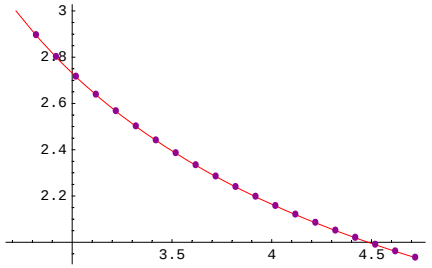
x	yRunge-Kutta 3
2.81828	2.89539
2.91828	2.80113
3.01828	2.71569
3.11828	2.63786
3.21828	2.56662
3.31828	2.50114
3.41828	2.44072
3.51828	2.38478
3.61828	2.33281
3.71828	2.28438
3.81828	2.23913
3.91828	2.19674
4.01828	2.15694
4.11828	2.11948
4.21828	2.08415
4.31828	2.05077
4.41828	2.01917
4.51828	1.98921
4.61828	1.96074
4.71828	1.93367

```
GRK3 = ListPlot[RK3, PlotStyle → {PointSize[0.015],
    RGBColor[0.572549, 0.0196078, 0.572549]}]
```



- Graphics -


```
Show[GPS, GRK3]
```



- Graphics -

absolútna chyba riešenia v koncovom bode intervalu x =2+e

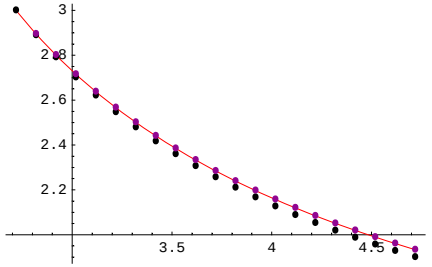
```
x[20]
```

4.71828

```
EpsRK3 = Abs[yps[x[20]] - yRK3[20]]
```

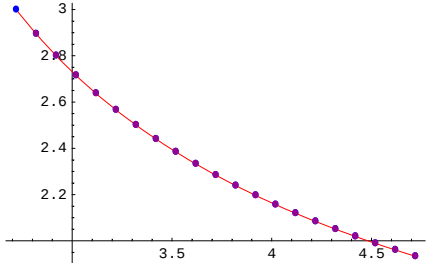
0.0000103718

```
Show[GPS, GE, GRK3]
```



- Graphics -

```
Show[GPS, GRK2, GRK3]
```



- Graphics -

f) RUNGE - KUTTA ORDER 4

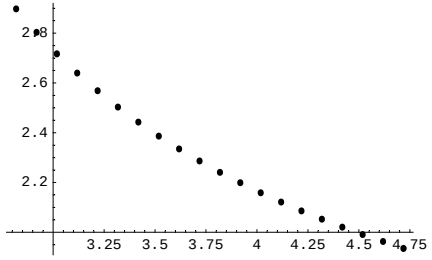
```
x[0] = E;
yRK4[0] = 3;
Do[
    k1 = f[x[i], yRK4[i]];
    k2 = f[x[i] + h / 2, yRK4[i] + h / 2 * k1];
    k3 = f[x[i] + h / 2, yRK4[i] + h / 2 * k2];
    k4 = f[x[i] + h, yRK4[i] + h * k3];
   yRK4[i + 1] = yRK4[i] + h / 6 * (k1 + 2 * k2 + 2 * k3 + k4);
  x[i + 1] = x[i] + h,
   {i, 0, iter}];
```

```
RK4 = Table[{x[i], yRK4[i]}, {i, 1, iter}];
TableForm[RK4, TableHeadings → {None, {"x", "yRunge-Kutta 4"}}]
```

Null[3]

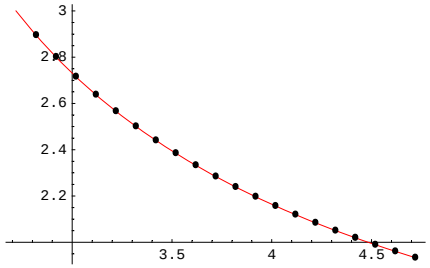
x	yRunge-Kutta 4
2.81828	2.8954
2.91828	2.80113
3.01828	2.7157
3.11828	2.63787
3.21828	2.56663
3.31828	2.50115
3.41828	2.44073
3.51828	2.38479
3.61828	2.33282
3.71828	2.28439
3.81828	2.23914
3.91828	2.19675
4.01828	2.15695
4.11828	2.11949
4.21828	2.08416
4.31828	2.05078
4.41828	2.01918
4.51828	1.98922
4.61828	1.96076
4.71828	1.93368

```
GRK4 = ListPlot[RK4, PlotStyle → {PointSize[0.015]}]
```



- Graphics -


```
Show[GPS, GRK4]
```



- Graphics -

absolútna chyba riešenia v koncovom bode intervalu x =2+e

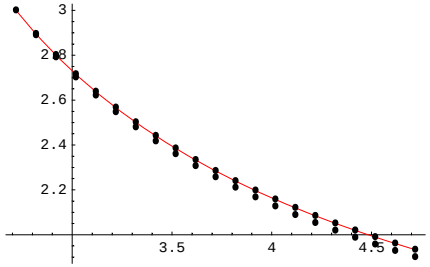
```
x[20]
```

4.71828

```
EpsRK4 = Abs[yps[x[20]] - yRK4[20]]
```

3.86186×10^{-8}

```
Show[GPS, GE, GRK4]
```



- Graphics -

g) Porovnanie riešení a globálnych chýb: presného, Eulerovho a riešer Runge-Kutta metód

```
Comp = Table[{i, x[i], yps[x[i]], yE[i], Abs[yps[x[i]] - yE[i]], yRK2[i], Abs[y
    yRK3[i], Abs[yps[x[i]] - yRK3[i]], yRK4[i], Abs[yps[x[i]] - yRK4[i]]}, {i,
TableForm[Comp, TableHeadings → {None, {"i", "x", "yExact", "yEuler", "ErrEul
    "yRunge2", "ErrRunge2", "yRunge3", "ErrRunge3", "yRunge4", "ErrRunge4"}}]
```

i	x	yExact	yEuler	ErrEuler	yRunge2	Er
0	e	3	3	0	3	0
1	2.81828	2.8954	2.88964	0.00576062	2.89553	0.
2	2.91828	2.80113	2.79068	0.0104541	2.80136	0.
3	3.01828	2.7157	2.70139	0.0143086	2.71601	0.
4	3.11828	2.63787	2.62037	0.0174961	2.63823	0.
5	3.21828	2.56663	2.54648	0.0201477	2.56704	0.
6	3.31828	2.50115	2.47879	0.0223649	2.5016	0.
7	3.41828	2.44073	2.41651	0.0242268	2.44121	0.
8	3.51828	2.38479	2.35899	0.0257962	2.38529	0.
9	3.61828	2.33282	2.30569	0.0271228	2.33334	0.
10	3.71828	2.28439	2.25614	0.0282468	2.28492	0.
11	3.81828	2.23914	2.20994	0.0292008	2.23968	0.
12	3.91828	2.19675	2.16674	0.0300112	2.1973	0.
13	4.01828	2.15695	2.12625	0.0307	2.1575	0.
14	4.11828	2.11949	2.0882	0.0312852	2.12005	0.
15	4.21828	2.08416	2.05238	0.0317817	2.08473	0.
16	4.31828	2.05078	2.01858	0.0322022	2.05135	0.
17	4.41828	2.01918	1.98662	0.0325571	2.01975	0.
18	4.51828	1.98922	1.95636	0.0328554	1.98978	0.
19	4.61828	1.96076	1.92765	0.0331048	1.96132	0.
20	4.71828	1.93368	1.90037	0.0333115	1.93425	0.

h) Na určenie numerického riešenia použite príkaz NDSolve.

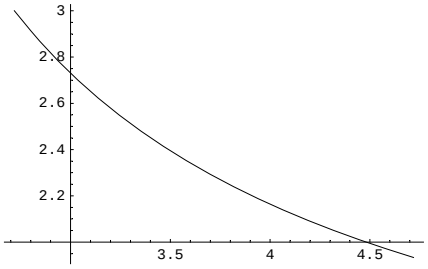
```
h = NDSolve[{y[x] + x * y'[x] * Log[x] == 0, y[E] == 3}, y[x], {x, E, 2 + E}]
```

```
{{y[x] → InterpolatingFunction[{{2.71828, 4.71828}}, <>][x]}}
```

```
y[x_] = h[[1, 1, 2]]
```

```
InterpolatingFunction[{{2.71828, 4.71828}}, <>][x]
```

```
Plot[y[x], {x, E, 2 + E}]
```




```
- Graphics -
```


g) Porovnanie riešení a globálnych chýb: presného, Eulerovho a riešer Runge-Kutta metód

```
ps[x[i]] - yRK2[i]],
0, iter}];
er",
]
```

rRunge2	yRunge3	ErrRunge3	yRunge4	ErrRunge4
	3	0	3	0
000130318	2.89539	3.00821×10^{-6}	2.8954	1.23252×10^{-8}
000230131	2.80113	5.17917×10^{-6}	2.80113	2.09835×10^{-8}
000307198	2.71569	6.75822×10^{-6}	2.7157	2.71085×10^{-8}
000367078	2.63786	7.91236×10^{-6}	2.63787	3.14555×10^{-8}
000413819	2.56662	8.75734×10^{-6}	2.56663	3.45373×10^{-8}
00045041	2.50114	9.3747×10^{-6}	2.50115	3.67082×10^{-8}
000479085	2.44072	9.82278×10^{-6}	2.44073	3.82164×10^{-8}
000501536	2.38478	0.0000101438	2.38479	3.92383×10^{-8}
000519058	2.33281	0.0000103688	2.33282	3.9901×10^{-8}
000532649	2.28438	0.0000105208	2.28439	4.02968×10^{-8}
000543089	2.23913	0.0000106168	2.23914	4.04935×10^{-8}
000550988	2.19674	0.00001067	2.19675	4.05415×10^{-8}
000556831	2.15694	0.0000106902	2.15695	4.04784×10^{-8}
000561003	2.11948	0.0000106851	2.11949	4.03323×10^{-8}
000563812	2.08415	0.0000106603	2.08416	4.01248×10^{-8}
000565509	2.05077	0.0000106206	2.05078	3.98721×10^{-8}
000566295	2.01917	0.0000105693	2.01918	3.95866×10^{-8}
000566335	1.98921	0.0000105094	1.98922	3.9278×10^{-8}
000565763	1.96074	0.000010443	1.96076	3.89534×10^{-8}
00056469	1.93367	0.0000103718	1.93368	3.86186×10^{-8}

h) Na určenie numerického riešenia použite príkaz NDSolve.

```
y[3.5]
```

```
2.39471
```

```
y[x] /. x → 3.5
```

```
2.39471
```

absolútna chyba riešenia v koncovom bode intervalu x =2+e

```
x[20]
```

```
4.71828
```

```
y[2 + E]
```

```
1.93368
```

```
EpsND = Abs[yps[x[20]] - y[2 + E]]
```

$$1.05227 \times 10^{-8}$$

```
i = 20
```

```
20
```

```
{{"x", "yRunge4", "ErrRunge4", "yND", "ErrND"},
  {2 + E, yRK4[i], Abs[yps[x[i]] - yRK4[i]], y[2 + E],
   Abs[yps[x[i]] - y[2 + E]]}} // TableForm
```

x	yRunge4	ErrRunge4	yND	ErrND
$2+e$	1.93368	3.86186×10^{-8}	1.93368	1.05227×10^{-8}

```
Clear[y]
```

Samostatná práca :

Riešte diferenciálnu rovnicu
$(x+1)y' = x^2 + y^2$, y(2) = -1,
pre $x \in \langle 2; 2{,}5 \rangle$ a krok h=0,05.

Čiastkové úlohy:
a) Vyšetrite existenciu a jednoznačnosť riešenia.
b) Ak je to možné, určte presné všeobecné a partikulárne riešenie. Použite príkaz DSolve.
c) Eulerovou metódou nájdite približné riešenie.
d) Nájdite približné riešenie metódou Runge-Kutta 2. rádu.
e) Nájdite približné riešenie metódou Runge-Kutta 3. rádu.
f) Nájdite približné riešenie metódou Runge-Kutta 4. rádu.
g) V koncovom bode intervalu x = 2,5 vypočítajte globálne chyby všetkých približných riešení.
h) Na určenie numerického riešenia použite príkaz NDSolve.