Indefinite Integrals,
Integration by Parts, The Substitution Method

Often occuring problem in many mathematical and technical applications is the problem, reverse to the differentiation: Find, to a given function $f(x)$, such a function $F(x)$, that $F'(x) = f(x)$.

If for each x from an interval $J : F'(x) = f(x)$, then the function $F(x)$ is said to be an **antiderivative** of $f(x)$ on *J*.

For example, the function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $J = (-\infty, \infty)$. But it is obvious that if *c* is any real number, then the function $G(x) = x^2 + c$ is an antiderivative of $f(x) = 2x$ as well. It holds:
If $F(x)$ is an antiderivative of $f(x)$ on an interval *J*, then a function $G(x)$ is antiderivative of $f(x)$ on *J* if and only if there exists such a real number *C*, that: $\forall x \in J : G(x) = F(x) + C$

Example 1. Find antiderivative for the function $f : y = 1 - 3x^2$, with the graph, passing through the point $[1, 2]$.

The set of all antiderivatives of a function $f(x)$ on an interval *J* is called the **indefinite integral** of *f* on *J* and the notation is $\int f(x)dx$. When a formula $F(x) + C$ gives all antiderivatives, we indicate this by writing: $\int f(x)dx = F(x) + C \quad (\forall x \in J)$
To integrate a function, it means to find all its antiderivatives, thus its indefinite integral. Operations of differentiation and integration are inverse to each other: $\int f'(x)dx = f(x) + C$, and $\left[\int f(x)dx\right]' = f(x)$ (for each constant).

If a function $f(x)$ is continuous on an open interval *J*, then it possesses an antiderivative on *J* (it is integrable on *J*).

From the definition and the formulas for derivatives follow basic integration formulas-tablet's integrals:

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C,\ a \neq -1,\ a \in R$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C,\ a > 0,\ a \neq 1 \Rightarrow \int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} dx = tgx + C$$

$$\int \frac{1}{\sin^2 x} dx = -\cot gx + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C, \, a > 0 \Rightarrow \int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C, \;\; a > 0 \Rightarrow \int \frac{1}{1 + x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

Each of these formulas is valid in any open interval, contained in the domain of definition of corresponding antiderivative.

From basic rules of differentiation it follows:

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int k \cdot f(x) dx = k \cdot \int f(x) dx, \, k \in R$$

These rules are valid in any open interval of integrability $f(x)$ and $g(x)$.

Example 2. Calculate: a) $\int \left(2x^5 - x^3 + 5\right) dx$, b) $\int \frac{5\sqrt[3]{x} + x^3 - x}{x} dx$, c) $\int \frac{3x}{x^2 + 1} dx$

Example 3. Let the velocity of a rectilinear motion is given by: $v(t) = t^2$ (t-time of motion). Find the law of the motion, $s = s(t)$ under the condition $s(0) = 1$

Integration by Parts

This method is most frequently used in the integration of expressions that may be represented in the form of a product of two functions $u(x)$ and $v(x)$ in such a way, that the finding of the function $u(x)$ and evaluation of the integral $\int u \cdot v' dx$ is a simpler problem than the direct evaluation of the original integral $\int u' \cdot v dx$:

Let $u(x)$ and $v(x)$ be functions possessing continuous derivatives. Then

$$\int u'(x) v(x) dx = u(x) \cdot v(x) - \int u(x) \cdot v'(x) dx$$

Example 4. Compute integrals: $\int xe^x dx$, $\int x^2 \cos x dx$, $\int \sin^2 x dx$, $\int arctgx dx$, $\int \ln x dx$, $\int \frac{\ln x}{x} dx$, $\int e^x \sin x dx$

The Substitution Method of Integration.

A change of variable can often change an unfamiliar integral into one we know how to evaluate. The method of doing this is called "The Substitution Method". It is (mostly) used, if integrand is a function of the form $f(\varphi(x)\cdot\varphi'(x))$; Let $\int f(u)du = F(u)+C$, on (α,β). Then

$$\int f(\varphi(x))\cdot\varphi'(x)dx = F(\varphi(x))+C \text{ on } (a,b)$$

In short: $\int f(\varphi(x))\cdot\varphi'(x)dx = \int f(u)du$, for $u=\varphi(x)$

Example 5. Compute integrals:

$\int 2x\cos(x^2+1)dx$, $\int \sin 5x dx$, $\int \cos^7 x \sin x dx$, $\int \sqrt{1+3x}dx$, $\int \sin^5 x dx$, $\int \frac{\ln^4 x}{x} dx$,

$\int \frac{x}{x^2+5} dx$, $\int \frac{arctg^2 x}{1+x^2} dx$, $\int \frac{\sin x}{2+\cos x} dx$, $\int 3xe^{x^2} dx$, $\int (3x+7)^5 dx$

In integrals in Example 5 we use a substitution $u=\varphi(x)$. But it is also possible to make a substitution by taking x as a function of $u : x=\varphi(u)$. In this case a suitable function φ should be chosen so that one can evaluate obtained indefinite integral and determine the inverse function φ^{-1}: $\int f(x)dx = \int f(\varphi(u))\cdot\varphi'(u)du, \quad u=\varphi^{-1}(x)$

Example 6. Compute integrals

a) $\int \sin\sqrt{x}dx$, b) $\int \sqrt{4-x^2}dx$

Generally, the integration process consists in transforming the given integral to an integral already known by means of algebraical transformation of the integrand, integration by parts and integration by change of variable.

Example 7. Compute integrals

$\int \arcsin x dx$, $\int (x^2+1)e^{-x} dx$, $\int x\cdot\arctan x dx$, $\int x\cdot\ln^2 x dx$, $\int \cos(\ln x)dx$,

$\int \frac{\ln(\ln x)}{x} dx$, $\int \frac{x\cdot\arcsin x}{\sqrt{1-x^2}} dx$, $\int \arctan\sqrt{x}dx$, $\int \frac{1}{x^2}\sin\frac{1}{x} dx$, $\int e^{\sqrt{x}} dx$