Global Extrema, Convexity and Concavity, Investigation of the Behaviour of a Function

Global Extrema (Absolute Extrema)

Let a function $f(x)$ be defined on a set M and let $x_0 \in M$. The value $f(x_0)$ is said to be a **global (absolute) maximum** of f on M, if: $\forall x \in M : f(x) \leq f(x_0)$.
The value $f(x_0)$ is said to be a **global (absolute) minimum** of f on M, if: $\forall x \in M : f(x) \geq f(x_0)$.

Therefore the global maximum is the greatest and the global minimum is the least value assumed by the function f (on M).

Speaking about minimum or maximum without specifying the set, it is considered to be the least or the greatest value of f on $D(f)$.

Often appearing problem is to find the greatest (maximum) and the least (minimum) values of a continuous function in a closed interval $\langle a,b \rangle$.

We proceed as follows:

1. We find all critical points $x_1, x_2, ...x_k \in (a,b)$
2. We compute $f(a), f(b), f(x_1), f(x_2), ... f(x_k)$

The greatest of these values is the global maximum of f in $\langle a,b \rangle$ and the least of these values is the global minimum of f in $\langle a,b \rangle$.

Example 1. Find the least and the greatest values of $f : y = 2x^3 - 3x^2 - 12x + 1$, in the interval $\langle -2,4 \rangle$.

Example 2. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have and what are its dimensions.

Example 3. A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence. With 800m of fence at your disposal, what is the largest area you can enclose?

Remark. If f fails to be continuous or M is not a closed interval, then minimum and maximum f on M can be, but need not be reached.

Example 4. Show that among all rectangles with a given perimeter P, the one with the largest area is a square.

Example 5. An open rectangular box is to be made from a piece of cardboard 8 in wide and 15 in long by cutting squares from the corners and folding up the sides. Find the dimensions of the box of largest volume.

Convexity, Concavity, Points of Inflection

Let $f(x)$ be a function differentiable on an interval J. The function f is called **convex** (**concave**) on J, if all points of its graph on J lie above (below) any tangent line to $G(f)$ on this interval (excepting point of tangency). Let f be continuous at a point x_0. If there exists $\varepsilon > 0$ such that f is concave (convex) in $N_\varepsilon^-(x_0)$ and convex (concave) in $N_\varepsilon^+(x_0)$, the point x_0 is called the **point of inflection** of $f(x)$.

The second Derivative Test for Concavity and Convexity:
If $f''(x) > 0$, for each $x \in J$, then $f(x)$ is convex on J,
if $f''(x) < 0$, for each $x \in J$, then $f(x)$ is concave on J.

It follows:
If $f(x)$ is continuous at x_0 and $f''(x) > 0$ ($f''(x) < 0$) in $N_\varepsilon^-(x_0)$ and $f''(x) < 0$ ($f''(x) > 0$) in $N_\varepsilon^+(x_0)$, then x_0 is a point of inflection.

Moreover: If x_0 is a point of inflection of f, then either $f''(x_0) = 0$ or $f''(x_0)$ doesn't exist.
If f is three times differentiable at a point x_0, $f''(x_0) = 0$ and $f'''(x) \neq 0$, then x_0 is a point of inflection.

Example 6. Find intervals of convexity and concavity and points of inflection for functions:

$f_1 : y = x^3 + 3x^2 - 9x + 5$

$f_2 : y = x^3$

$f_3 : y = x^4$

$f_4 : y = \ln\dfrac{1+x}{1-x}$

$f_5 : y = x - \sin x$

$f_6 : y = \sqrt[3]{x}$

$f_7 : y = \left|x^2 - 1\right|$

General Scheme for the Investigation of the Graph of a Function.

For a given function $f(x)$ this scheme is divided to six stages at which we determine, in succession, the following elements of the behaviour of the function:

1. $D(f)$, points of discontinuity, zero points (the points at which $G(f)$ cuts the axes of coordinates)
2. The character of symmetry of $G(f)$, that is whether the function is even or odd (or neither) and periodicity

3. All kinds of asymptotes (vertical, horizontal, inclined)
4. Intervals of monotonicity and points of extrema (local)
5. Intervals of convexity and concavity and points of inflection
6. $G(f), R(f)$ (if need be limits at improper points)

<u>Example 7.</u> Investigate the behaviour of functions

a) $f: y = \dfrac{x^2+1}{3x}$

b) $f: y = \dfrac{2-3x}{x-2}$

c) $f: y = e^{-x^2}$

d) $f: y = \dfrac{\ln x}{x}$ $\quad f: y = x - 2\arctan x$

e)

f) $f: y = \ln\left(4-x^2\right)$

g) $f: y = x - \dfrac{1}{x}$

h) $f: y = x + \dfrac{1}{x}$

i) $f: y = \dfrac{1}{x^2-4}$

j) $f: y = \dfrac{x}{x^2+4}$

k) $f: y = \dfrac{x^3}{2(x+1)^2}$

l) $f: y = x^2 + \dfrac{1}{x^2}$

m) $f: y = x^2 \cdot e^{-x}$

<u>Example 8</u>. What is the largest area possible for a rectangle inscribed into a semicircle of radius 2?

<u>Example 9</u>. Find maximal area of an isosceles triangle inscribed into a circle with the radius $R > 0$.