L'Hospital's Rule
Monotonicity, Local Extrema

L'Hospital's Rule

Suppose:

a) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, or $\lim\limits_{x\to 0} |f(x)| = \lim\limits_{x\to a} |g(x)| = \infty$

b) There exists $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ (proper or improper).

Then there exists $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ and it holds: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$. This Rule is valid also for limits at improper points and for one-sided limits.

<u>Example 1</u>. By means of L'Hospital's Rule compute: $\lim\limits_{x\to 2} \dfrac{x^2-5x+6}{x^2-3x+2}$, $\lim\limits_{x\to 0} \dfrac{3^x-1}{\sin x}$, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$, $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$, $\lim\limits_{x\to\infty} \dfrac{5x^2-3x}{7x^2+1}$, $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$, $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{2x-\pi}{\cos x}$, $\lim\limits_{x\to \frac{\pi}{2}^+} \dfrac{\tan x}{1+\tan x}$

The indeterminate limit forms $\pm\infty\cdot 0$ and $\infty-\infty$ can sometimes be handled by changing the expressions to the forms $0/0$ or ∞/∞.

<u>Example 2.</u> Find limits:

a) $\lim\limits_{x\to\frac{\pi}{2}} \left(\tan x - \dfrac{1}{\cos x}\right)$, $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$, $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sqrt{x}}\right)$

b) $\lim\limits_{x\to 0^+} x\cdot\ln x$, $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x}$, $\lim\limits_{x\to 1^-} \dfrac{1}{1-x}\cdot\ln x$

Example 3. By means of L'Hospital's Rule prove:

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

Monotonicity

The application of the differential calculus to the investigation of functions is based on a simple relationship between the behaviour of a function and properties of its derivatives and, particularly, of the first derivative. Increase is associated with positive derivatives and decrease with negative derivatives:

Suppose that a function $f(x)$ is differentiable at every point x of an interval J. Then

1. f is increasing on J if $f'(x) > 0, \forall x \in J$
2. f is decreasing on J if $f'(x) < 0, \forall x \in J$
3. f is non-decreasing on J if $f'(x) \geq 0, \forall x \in J$
4. f is non-increasing on J if $f'(x) \leq 0, \forall x \in J$

In geometric terms: Differentiable functions increase on intervals where their graphs have position slopes and decrease on intervals where their graphs have negative slopes.

<u>Example 4.</u> Show, that the function $f: y = \log_{0.5} x$ is decreasing on $D(f)$ and the function $g: y = x^3 - 2x^2 + 5x$ is increasing on $D(g)$.

<u>Example 5.</u> Find intervals of monotonicity for the functions: $f_1: y = x^3 - 3x$ and $f_2: y = x^4 - 4x^3 + 4x^2$

Geometrically, it appears evident that if the derivative of a function takes zero values at some separate points but retains constant sign at all other points, the function is also strictly monotone (increasing or decreasing) in the given interval.

For instance, the function $f: y = x - \sin x$ is increasing and $f'(2k\pi) = 0, \forall k = 0, \pm 1, \pm 2, ...$

<u>Example 6.</u> With the aid of first derivative show, that the equation $x - \cos x = 0$ has one and only one root in the interval $\left\langle 0, \frac{\pi}{2} \right\rangle$.

Local (Relative) Extrema

Let a function $f(x)$ be defined in a neighbourhood of a point x_0. The value $f(x_0)$ is said to be a **local maximum** of $f(x)$, if there exists such an $N_\varepsilon(x_0)$, that $f(x) \leq f(x_0), \forall x \in N_\varepsilon(x_0)$. Then x_0 is called the **point of local maximum**.

The value $f(x_0)$ is said to be a **local minimum** of $f(x)$, if there exists such an $N_\varepsilon(x_0)$, that $f(x) \geq f(x_0), \forall x \in N_\varepsilon(x_0)$. Then x_0 is called the **point of local minimum**.

If $f(x) < f(x_0)$ or $f(x) > f(x_0)$ for $\forall x \in N_\varepsilon(x_0), x \neq x_0$, the value $f(x_0)$ is called the **strict** local maximum or minimum, resp.
When speaking of a maximum or a minimum at a point, we usually mean a strict extremum.

If x_0 is point of local extremum of a function f and f is differentiable at x_0, then $f'(x_0) = 0$.
It follows that a function can posses local extrema at points at which the derivative is equal zero (these are called **stationary points**) or at points at which the derivative doesn't exist.

The points where $f'=0$ or fails to exist are commonly called the critical points of f (for the first derivative).

Example 7. Find all points of local extrema for functions: $f_1 : y = \cos x$, $f_2 : y = x^3$, $f_3 : y = |x-1|$

If a function f is differentiable at each $x \in N_\varepsilon(x_0), x \neq x_0$ and $f'(x) > 0$ $(f'(x) < 0)$ for $\forall x \in (x_0 - \varepsilon, x_0)$ and $f'(x) < 0$ $(f'(x) > 0)$ for $\forall x \in (x_0, x_0 + \varepsilon)$ then $f(x_0)$ is strict local maximum (minimum).

It means that for a local maximum $f(x_0)$ f is increasing in $N_\varepsilon^-(x_0)$ and decreasing in $N_\varepsilon^+(x_0)$ and for a local minimum $f(x_0)$, f is decreasing in $N_\varepsilon^-(x_0)$ and increasing in $N_\varepsilon^+(x_0)$.

The second derivative test:

If $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then x_0 is a point of local extremum. If $f''(x_0) < 0$, then $f(x_0)$ is the strict local maximum and if $f''(x_0) > 0$, then $f(x_0)$ is the strict local minimum.

Example 8. Find all local extrema for functions: $f_1 : y = x^4 - 4x^3 + 4x^2$, $f_2 : y = x^4$, $f_3 : y - \sqrt[3]{x^2}$

Example 9. Find two positive numbers whose sum is 20 and whose product is as large as possible.

Example 10. We have to make a can with a given volume $V > 0$, shaped like a right circular cylinder. What dimensions will use the least material.